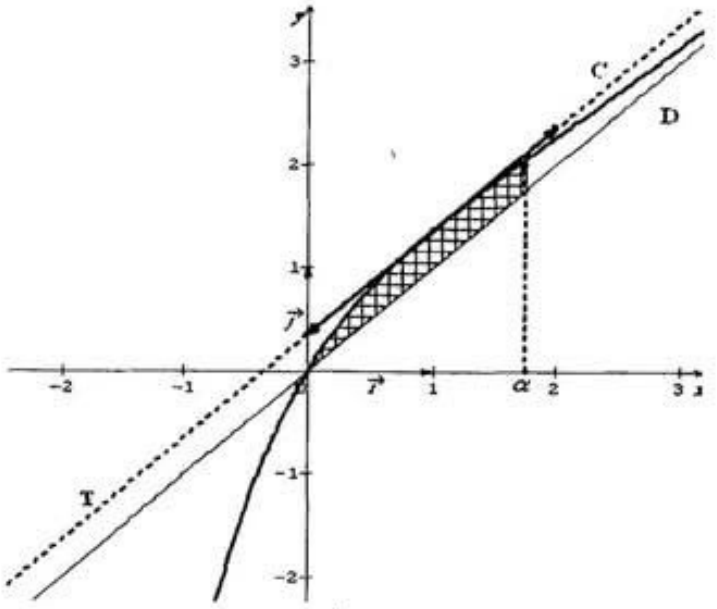

3) $\alpha \in \mathbb{R}_+^*$

*/ $A(\alpha) = \int_0^\alpha |f(x) - x| dx = \left(\int_0^\alpha xe^{-x} dx \right)$

Calculons $A(\alpha)$ par parties :

$U(x) = x \rightarrow U'(x) = 1$

$V'(x) = e^{-x} \rightarrow V(x) = -e^{-x}$

D'où :

$$A(\alpha) = \left[-xe^{-x} \right]_0^\alpha - \int_0^\alpha -e^{-x} dx$$

$$A(\alpha) = \left(\left(-\alpha e^{-\alpha} \right) - \left[e^{-x} \right]_0^\alpha \right) \cdot \|\vec{i}\| \cdot \|\vec{j}\|$$

$$\boxed{A(\alpha) = \left(-\alpha e^{-\alpha} - e^{-\alpha} + 1 \right) \cdot \|\vec{i}\| \cdot \|\vec{j}\|}$$

*/

$$\lim_{\alpha \to +\infty} -\alpha e^{-\alpha} - e^{-\alpha} + 1 ; \text{ on pose } t = -\alpha$$

$$= \lim_{t \to -\infty} te^t - e^t + 1$$

$$= 1$$

Donc $\boxed{\lim_{\alpha \to +\infty} -\alpha e^{-\alpha} - e^{-\alpha} + 1 = 1}$

4) Pour $x < 1$ on a :

$$h(x) = \ln\left(g(x) - 1 \right)$$

a) pour $x < 1$:

*/ $h(x) = \ln\left[(1-x)e^{-x} + 1 - 1 \right]$

$= \ln\left[(1-x)e^{-x} \right]$; $e^{-x} > 0, 1 - x > 0$

$= \ln(1-x) + \ln e^{-x}$

$= \ln(1-x) - x$

Donc $\forall x < 1$ $\boxed{h(x) = \ln(1-x) - x}$

*/ $1 - \dfrac{1}{1-x} = \dfrac{1-x}{1-x} - \dfrac{1}{1-x}$

$= \dfrac{1-x-1}{1-x} = \dfrac{-x}{1-x}$

D'où $\boxed{1 - \dfrac{1}{1-x} = \dfrac{-x}{1-x}}$

b) $I = \int_{-1}^0 \ln(1-t)\, dt$

Intégrant par parties :

$U(t) = \ln(1-t) \rightarrow U'(t) = \dfrac{-1}{1-t}$

$V'(t) = 1 \rightarrow V(t) = t$

D'où :

$$I = \left[t \ln(1-t) \right]_{-1}^0 - \int_{-1}^0 \frac{-t}{1-t} dt$$

$$I = 0 - (-1)\ln 2 - \int_{-1}^0 \left(1 + \frac{-1}{1-t} \right) dt$$

$$I = \ln 2 - \left[t + \ln(1-t) \right]_{-1}^0$$

$$I = \ln 2 - \left[0 - (-1 + \ln 2) \right]$$

$$I = \ln 2 - 1 + \ln 2$$

$$I = 2\ln 2 - 1$$

Donc : $\boxed{I = 2\ln 2 - 1}$

c)

$$J = \int_{-1}^0 h(t)\, dt = \int_{-1}^0 \left[-t + \ln(1-t) \right] dt$$

$$= \int_{-1}^0 -t\, dt + \int_{-1}^0 \ln(1-t)\, dt = \left[-\frac{t^2}{2} \right]_{-1}^0 + I$$

$$= \left[0 - \left(-\frac{(-1)^2}{2} \right) \right] + I = \frac{1}{2} + 2\ln 2 - 1$$

$$= 2\ln 2 - \frac{1}{2} = J$$